Expansions of relative operator entropies and operator valued $alpha$-divergence (Theory of operator means and related topics)

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1. Introduction

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. An operator T on H is said to be positive (denoted by $T\geq 0$) if $(Tx,x)\geq 0$ for all $x\in H$, and an operator T is said to be strictly positive (denoted by $T>0$) if T is invertible and positive.

For strictly positive operators A and B, and for $x\in\mathbf{R}$, a path passing through A and B is defined as follows ([4], [5], [11] etc.):

$$A\natural_{x}B\equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{x}A^{\frac{1}{2}}.$$

If $x\in[0,1]$, then the path becomes weighted geometric operator mean denoted by $A\ \sharp_{x}\ B$. Weighted arithmetic operator mean is defined as $A\ \nabla_{x}\ B\equiv(1-x)A+xB$ for $x\in[0,1]$.

Fujii and Kamei [3] defined relative operator entropy as follows:

$$S(A|B)\equiv A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

Furuta [8] gave a generalized form of relative operator entropy. Furuta's one is called generalized relative operator entropy and is defined as follows:

$$S_{t}(A|B)\equiv A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{t}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},\ t\in\mathbf{R}.$$

We remark that $S_{0}(A|B)=S(A|B)$ holds.

Yanagi, Kuriyama and Furuichi [13] defined Tsallis relative operator entropy as follows:

$$T_{t}(A|B)=\frac{A\ \sharp_{t}\ B-A}{t},\ t\in(0,1].$$

By replacing $A\ \sharp_{t}\ B$ with $A\ \natural_{t}\ B$, Tsallis relative operator entropy can be extended as the notion for $t\in\mathbf{R}$. Since $\lim_{t\to 0}\frac{a^{t}-1}{t}=\log a$ holds for $a>0$, we have $T_{0}(A|B)\equiv\lim_{t\to 0}T_{t}(A|B)=S(A|B)$.

For these relative operator entropies, we can give geometrical interpretations. By the derivative of the path with respect to x at t, we can get generalized relative operator entropy, that is, the following holds:

$$\left.\frac{d}{dx}A\natural_{x}B\right|_{x=t}=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{t}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\ =\ S_{t}(A|B).$$

Therefore, we can regard $S_{t}(A|B)$ as the slope of the tangent line at $x=t$ of the path. From this interpretation, we can regard $S(A|B)$ as the slope of the tangent line at $x=0$ of the path. Tsallis relative operator entropy can be regarded as the slope of the line passing through points A and $A\ \natural_{t}\ B$ on the path.

Amari [1] defined α-divergence as a notion to measure the difference between two probability distributions. Based on this notion, Fujii [2] defined operator valued α-divergence as follows: For strictly positive operators A and B, and for $\alpha \in (0,1)$,

$$D_\alpha(A|B) \equiv \frac{A \nabla_\alpha B - A \sharp_\alpha B}{\alpha(1-\alpha)}.$$

Petz [12] introduced the operator divergence $D_{FK}(A|B) \equiv B - A - S(A|B)$. Fujii et al. [6, 7] showed the following relation between $D_{FK}(A|B)$ and operator valued α-divergences at end points for interval $(0,1)$.

$$\begin{aligned} D_0(A|B) &\equiv \lim_{t\to 0} D_t(A|B) = B - A - S(A|B), \\ D_1(A|B) &\equiv \lim_{t\to 1} D_t(A|B) = A - B + S_1(A|B). \end{aligned}$$

For the quantity $D_0(A|B)$, we give a geometrical interpretation shown in Figure 1.

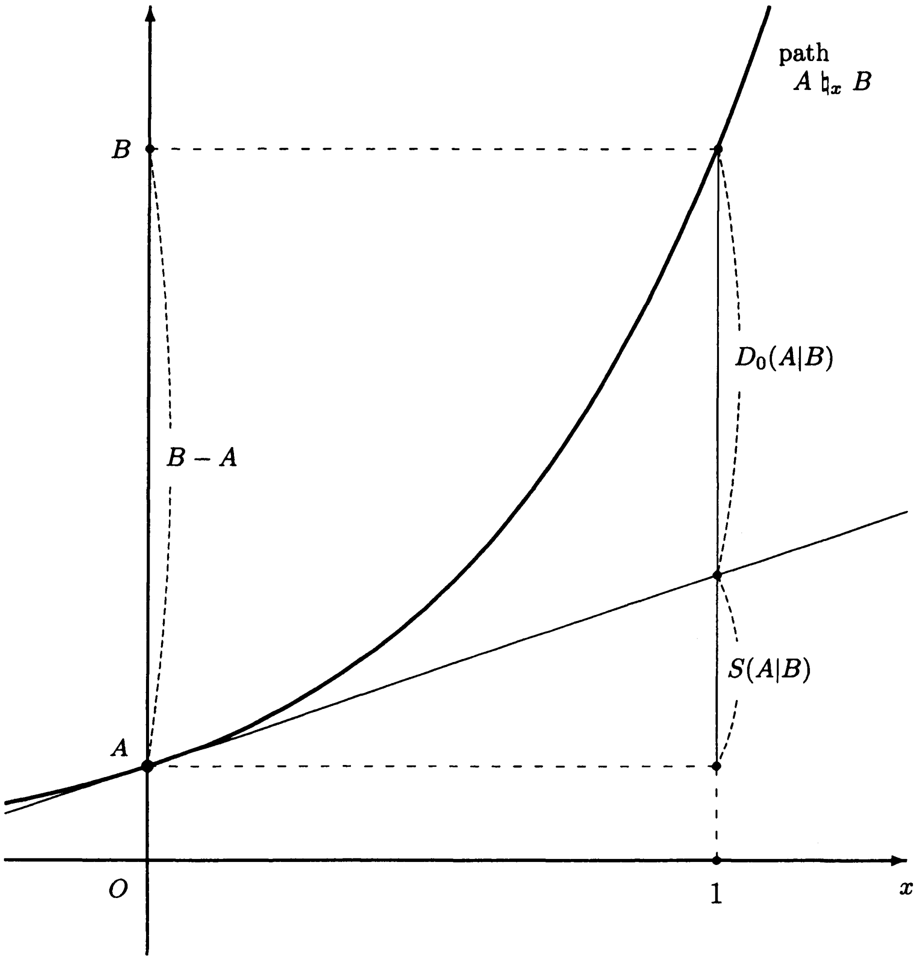

Figure 1: An interpretation for $D_0(A|B) = B - A - S(A|B)$.

In [11], Kamei showed that relative operator entropy has some kind of additivity as follows: For strictly positive operators A and B, and for $s \in \mathbf{R}$,

$$(\star) \qquad S(A|A \natural_s B) = sS(A|B).$$

In [9], we gave a viewpoint of operator valued distance for $S(A|B)$. Here, we give the following geometrical interpretation for this relation: The second component $A \natural_s B$ of the left hand side is an arbitrary point on the path. So, we can regard the relation $(\star)$ as relative operator entropy for a fixed point A and any point on the path.

For strictly positive operators A and B, and for $t \in [0,1]$ and $r \in [-1,1]$, operator power mean is defined as follows:

$$A \sharp_{t,r} B \equiv A^{\frac{1}{2}} \left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}} A^{\frac{1}{2}} = A \natural_{\frac{1}{r}} \{A \nabla_t (A \natural_r B)\}.$$

To preserve $(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r \geq 0$, we have to impose t in $[0,1]$. Operator power mean compounds the arithmetic, geometric and harmonic means, that is, the following holds.

arithmetic operator mean
$A \nabla_t B = (1-t)A + tB$

$\uparrow_{r=1}$

$A \sharp_{t,r} B \xrightarrow[r \to 0]{}$ geometric operator mean
$A \sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}$

$\downarrow_{r=-1}$

harmonic operator mean
$A \Delta_t B = (A^{-1} \nabla_t B^{-1})^{-1}$

We treat this operator power mean as an expanded path which links point A with point B. As the corresponding notions to relative operator entropies and operator valued α-divergence, we introduce expanded relative operator entropy, expanded Tsallis relative operator entropy and expanded operator valued α-divergence.

In this report, we aim at getting results on these expanded notions. In section 2, we show some results on relative operator entropies for two points on the path $A \natural_x B$. In section 3, we show results on expanded relative operator entropies, and in section 4, we show results on expanded operator valued α-divergence.

2. Relative operator entropies

Based on the relation $(\star)$, we show some basic results on relative operator entropies for two points on the path $A \natural_x B$.

To show the results in this section, we prepare the following properties of the path.

Lemma 2.1. *Let A and B be strictly positive operators. Then,*

$$A \natural_t (A \natural_s B) = A \natural_{st} B, \tag{1}$$
$$(A \natural_t B) \natural_s A = A \natural_{(1-s)t} B \tag{2}$$

hold for $s,\ t \in \mathbf{R}$.

Lemma 2.2. (Lemma 3.6 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A \natural_u B) \natural_w (A \natural_{u+v} B) = A \natural_{u+vw} B$$

holds for $u,\ v,\ w \in \mathbf{R}$.

For generalized relative operator entropy and Tsallis relative operator entropy, we have the following result corresponding to the relation $(\star)$.

Theorem 2.3. *Let A and B be strictly positive operators. Then,*

$$\begin{array}{lrcl} (1) & S_t(A|A \natural_s B) & = & sS_{st}(A|B), \\ (2) & T_t(A|A \natural_s B) & = & sT_{st}(A|B) \end{array}$$

hold for $s,\ t \in \mathbf{R}$.

Proof. (1) If $s = 0$, then it is obvious that the both sides equal zero, and if $t = 0$, then this equality becomes the relation $(\star)$. Otherwise, we get

$$\begin{aligned} & S_t(A|A \natural_s B) \\ = \ & A^{\frac{1}{2}} \left\{A^{-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}A^{-\frac{1}{2}}\right\}^t \log\left\{A^{-\frac{1}{2}}A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}A^{-\frac{1}{2}}\right\} A^{\frac{1}{2}} \\ = \ & A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{st} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}} \ = \ sS_{st}(A|B). \end{aligned}$$

(2) By (1) in Lemma 2.1, we have

$$T_t(A|A \natural_s B) \ = \ \frac{A \natural_t (A \natural_s B) - A}{t} \ = \ s\,\frac{A \natural_{st} B - A}{st} \ = \ sT_{st}(A|B).$$

$\square$

For the relation (1) in Theorem 2.3, we can give a geometrical interpretation shown in Figure 2. We remark that two tangent lines drawn in this figure intersect on the axis of the vertical direction.

In [10], we showed the following result on translation of generalized relative operator entropy.

Proposition 2.4. (Proposition 3.1 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = (A \natural_{u+v} B)(A \natural_u B)^{-1} S_u(A|B)$$

holds for $u,\ v \in \mathbf{R}$.

When we regard $S_u(A|B)$ and $S_{u+v}(A|B)$ as tangent vectors at u and $u + v$ on the path $A \natural_w B$, respectively, Proposition 2.4 means that $S_{u+v}(A|B)$ is parallelly transferring $S_u(A|B)$ by v along the path.

Here, we define the following noncommutative ratio on the path $A \natural_w B$.

Definition 2.5. *For strictly positive operators A and B, and for $u,\ v \in \mathbf{R}$, noncommutative ratio on the path $A \natural_w B$ is defined as follows:*

$$\mathcal{R}(u,v;A,B) \equiv (A \natural_{u+v} B)(A \natural_u B)^{-1}.$$

For the noncommutative ratio, the following property holds.

Proposition 2.6. (Proposition 3.3 in [10]) *Let A and B be strictly positive operators. Then,*

$$(A \natural_{u+v} B)(A \natural_u B)^{-1} = (A \natural_v B)A^{-1},$$

that is,

$$\mathcal{R}(u,v;A,B) = \mathcal{R}(0,v;A,B) = (A \natural_v B)A^{-1}$$

holds for $u,\ v \in \mathbf{R}$.

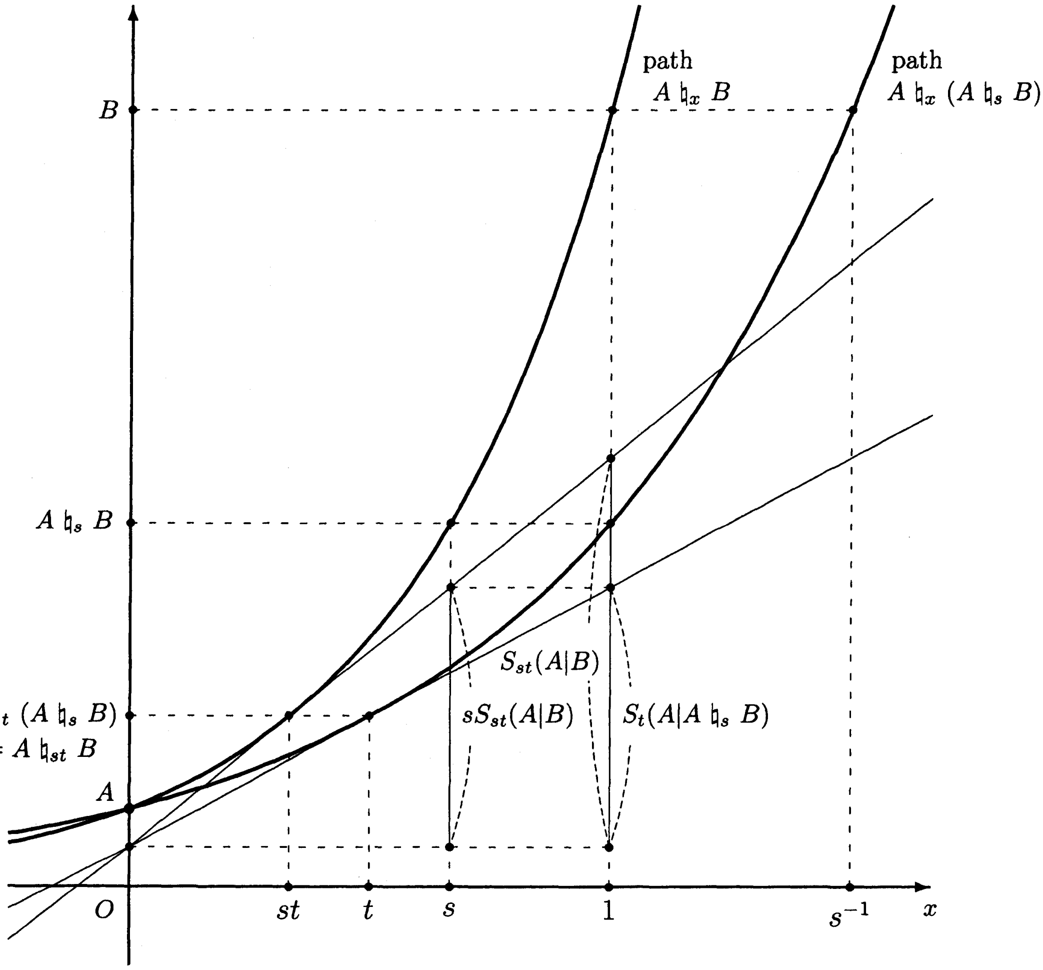

Figure 2: An interpretation for $S_t(A|A \natural_s B) = sS_{st}(A|B)$.

By Proposition 2.6, $\mathcal{R}(u, v; A, B)$ does not depend on u. So, we denote $\mathcal{R}(u, v; A, B)$ by $\mathcal{R}(v)$ in the rest of this report simply. We call multiplying by $\mathcal{R}(v)$ from the left side noncommutative ratio translation.

From Proposition 2.4 and Definition 2.5, we get the following immediately.

Corollary 2.7. (Corollary 3.4 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_{u+v}(A|B) = \mathcal{R}(v)S_u(A|B)$$

holds for u, $v \in \mathbf{R}$.

Remark 1. *By putting $u = 0$ in Corollary 2.7, we have $S_v(A|B) = \mathcal{R}(v)S(A|B)$.*

The following is an extension of the relation ($\star$). This is a result of generalized relative operator entropy for any two points on the path.

Proposition 2.8. (Proposition 3.7 in [10]) *Let A and B be strictly positive operators. Then,*

$$S_t(A \natural_v B|A \natural_{v+s} B) = sS_{v+st}(A|B)$$

hold for $s,\ t,\ v \in \mathbf{R}$.

By using noncommutative ratio, we can represent the results of relative operator entropies for any two points on the path as follows:

Theorem 2.9. (Theorem 3.11 in [10]) *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{ll}
(1) & S_t(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v) S_{st}(A|B), \\
(2) & T_t(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v) T_{st}(A|B), \\
(3) & S(A \natural_v B | A \natural_{v+s} B) = s\mathcal{R}(v) S(A|B)
\end{array}
$$

hold for $s,\ t,\ v \in \mathbf{R}$.

Proof. **(1)** By Proposition 2.8 and Corollary 2.7, we have

$$S_t(A \natural_v B | A \natural_{v+s} B) \;=\; sS_{v+st}(A|B) \;=\; s\mathcal{R}(v) S_{st}(A|B).$$

(2) By Lemma 2.2 and Proposition 2.6, we get

$$
\begin{aligned}
T_t(A \natural_v B | A \natural_{v+s} B) &= \frac{(A \natural_v B) \natural_t (A \natural_{v+s} B) - A \natural_v B}{t} \\
= \frac{A \natural_{v+st} B - A \natural_v B}{t} &= s\frac{(A \natural_v B)A^{-1}(A \natural_{st} B) - (A \natural_v B)A^{-1}A}{st} \\
= s(A \natural_v B)A^{-1} T_{st}(A|B) &= s\mathcal{R}(v) T_{st}(A|B).
\end{aligned}
$$

(3) This equality can be obtained by putting $t = 0$ for (1).

□

Remark 2. *We can get Theorem 2.3 by putting $v = 0$ for the relations (1) and (2) in Theorem 2.9.*

3. Expanded relative operator entropies

In this section, we show the results of expanded relative operator entropies for two points on the expanded path $A \sharp_{t,r} B$. Similarly to $S_t(A|B)$, in [9], expanded relative operator entropy $S_{t,r}(A|B)$ is defined by the derivative of expanded path with respect to x at t as follows: For strictly positive operators A and B, and for $t \in [0,1]$ and $r \in [-1,1]$,

$$
\begin{aligned}
S_{t,r}(A|B) &\equiv \left. \frac{d}{dx} A \sharp_{x,r} B \right|_{x=t} \\
&= A^{\frac{1}{2}} \left[\left\{ (1-t)I + t\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^r \right\}^{\frac{1}{r}-1} \frac{\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^r - I}{r} \right] A^{\frac{1}{2}} \\
&= \left[A \natural_{\frac{1}{r}-1} \{ A \nabla_t (A \natural_r B) \} \right] A^{-1} T_r(A|B).
\end{aligned}
$$

We remark that expanded relative operator entropy has the following relations [9]:

$$\begin{array}{ccccc} & & B-A & & \\ & & \uparrow_{r=1} & & \\ T_r(A|B) & \underset{t=0}{\longleftarrow} & S_{t,r}(A|B) & \underset{t=1}{\longrightarrow} & -T_r(B|A) \\ & & \downarrow_{r\to 0} & & \\ & & S_t(A|B) & & \end{array}$$

By replacing weighted geometric operator mean with operator power mean, we obtain the definition of expanded Tsallis relative operator entropy [9]: For strictly positive operators A and B, and for $t \in (0,1]$ and $r \in [-1,1]$,

$$T_{t,r}(A|B) \equiv \frac{A \sharp_{t,r} B - A}{t}.$$

We remark that expanded Tsallis relative operator entropy also has the following relations:

$$\begin{array}{ccccc} & & B-A & & \\ & & \uparrow_{r=1} & & \\ T_r(A|B) & \underset{t\to 0}{\longleftarrow} & T_{t,r}(A|B) & \underset{t=1}{\longrightarrow} & B-A \\ & & \downarrow_{r\to 0} & & \\ & & T_t(A|B) & & \end{array}$$

For expanded relative operator entropy, we can show the following result corresponding to (1) in Theorem 2.3.

Theorem 3.1. *Let A and B be strictly positive operators. Then,*

$$S_{t,r}(A|A \sharp_{s,r} B) = s(A \sharp_{st,r} B)\,\{A \nabla_{st} (A \natural_r B)\}^{-1} T_r(A|B) = sS_{st,r}(A|B)$$

holds for $t,\ s \in [0,1]$ and $r \in [-1,1]$.

In cases of $t \in \{0,\ 1\}$, we get the following relations.

Corollary 3.2. *Let A and B be strictly positive operators. Then,*

$$\begin{array}{llll} (1) & S_{0,r}(A|A \sharp_{s,r} B) & = & T_r(A|A \sharp_{s,r} B) \ = \ sT_r(A|B), \\ (2) & S_{1,r}(A|A \sharp_{s,r} B) & = & -T_r(A \sharp_{s,r} B|A) \\ & & = & s(A \sharp_{s,r} B)\,\{A \nabla_s (A \natural_r B)\}^{-1} T_r(A|B) \end{array}$$

hold for $s \in [0,1]$ and $r \in [-1,1]$.

To prove Theorem 3.1, we prepare the following lemmas. We omit their proofs.

Lemma 3.3. *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{llcl}
(1) & A \natural_r (A \sharp_{t,r} B) & = & A \nabla_t (A \natural_r B), \\
(2) & A \sharp_{t,r} (A \sharp_{s,r} B) & = & A \sharp_{st,r} B, \\
(3) & A \sharp_{t,r} B & = & B \sharp_{1-t,r} A
\end{array}
$$

hold for $t,\ s \in [0,1]$ and $r \in [-1,1]$.

Lemma 3.4. *Let A and B be strictly positive operators. Then,*

$$
\begin{aligned}
S_{t,r}(A|B) &= (A \sharp_{t,r} B) \{A \nabla_t (A \natural_r B)\}^{-1} T_r(A|B) \\
&= \left[A \natural_{\frac{1}{r}} \{A \nabla_t (A \natural_r B)\}\right] \{A \nabla_t (A \natural_r B)\}^{-1} T_r(A|B)
\end{aligned}
$$

holds for $t \in [0,1]$ and $r \in [-1,1]$.

Proof of Theorem 3.1. By Lemma 3.3 and Lemma 3.4, the following holds:

$$
\begin{aligned}
S_{t,r}(A|A \sharp_{s,r} B) &= \{A \sharp_{t,r} (A \sharp_{s,r} B)\} [A \nabla_t \{A \natural_r (A \sharp_{s,r} B)\}]^{-1} T_r(A|A \sharp_{s,r} B) \\
&= (A \sharp_{st,r} B) [A \nabla_t \{A \nabla_s (A \natural_r B)\}]^{-1} T_r(A|A \sharp_{s,r} B) \\
&= (A \sharp_{st,r} B) \{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{A \natural_r (A \sharp_{s,r} B) - A}{r} \\
&= (A \sharp_{st,r} B) \{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{A \nabla_s (A \natural_r B) - A}{r} \\
&= (A \sharp_{st,r} B) \{A \nabla_{st} (A \natural_r B)\}^{-1} \frac{(1-s)A + s(A \natural_r B) - A}{r} \\
&= s(A \sharp_{st,r} B) \{A \nabla_{st} (A \natural_r B)\}^{-1} T_r(A|B) = s S_{st,r}(A|B).
\end{aligned}
$$

□

The following (1) in Theorem 3.5 is a corresponding result to (2) in Theorem 2.3.

Theorem 3.5. *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{llcll}
(1) & T_{t,r}(A|A \sharp_{s,r} B) & = & s T_{st,r}(A|B) & (t \in (0,1]), \\
(2) & T_{0,r}(A|A \sharp_{s,r} B) & = & s T_r(A|B), & \\
(3) & T_{1,r}(A|A \sharp_{s,r} B) & = & s T_{s,r}(A|B), & \\
(4) & T_{1-t,r}(A \sharp_{s,r} B|A) & = & s \dfrac{t T_{st,r}(A|B) - T_{s,r}(A|B)}{1-t} & (t \in [0,1))
\end{array}
$$

hold for $s \in [0,1]$ and $r \in [-1,1]$.

Proof. (1) By (2) in Lemma 3.3, these can be shown as follows:

$$
T_{t,r}(A|A \sharp_{s,r} B) = \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A}{t} = s \frac{A \sharp_{st,r} B - A}{st} = s T_{st,r}(A|B).
$$

(2), (3) For (1), by putting $t = 0$ and $t = 1$, we can get these relations, respectively.

(4) Since $A \sharp_{t,r} B = B \sharp_{1-t,r} A$ holds for $t \in [0,1]$ and $r \in [-1,1]$, we have

$$
\begin{aligned}
T_{1-t,r}(A \sharp_{s,r} B|A) &= \frac{(A \sharp_{s,r} B) \sharp_{1-t,r} A - A \sharp_{s,r} B}{1-t} \\
&= \frac{A \sharp_{t,r} (A \sharp_{s,r} B) - A \sharp_{s,r} B}{1-t} = \frac{A \sharp_{st,r} B - A \sharp_{s,r} B}{1-t} \\
&= \frac{(A \sharp_{st,r} B - A) - (A \sharp_{s,r} B - A)}{1-t} = s \frac{t T_{st,r}(A|B) - T_{s,r}(A|B)}{1-t}.
\end{aligned}
$$

□

4. Expanded operator valued α-divergence

By Theorem 2.5 in [10] and the results in [6, 7], the following relation between operator valued α-divergence and Tsallis relative operator entropy was shown.

Theorem 4.1. ([6, 7, 10])*Let A and B be strictly positive operators. Then,*

$$D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$$

holds for $t \in [0,1]$.

Theorem 4.1 gives a geometrical interpretation for operator valued α-divergence. Tsallis relative operator entropy $T_t(A|B)$ can be regarded as the slope of the line passing through points A and $A \natural_t B$. Since $-T_{1-t}(B|A) = -\frac{B \natural_{1-t} A - B}{1-t} = \frac{B - A \natural_t B}{1-t}$, we can regard this operator value as the slope of the line passing through points $A \natural_t B$ and B. Therefore, $D_t(A|B)$ gives the difference between the slops of these two lines. We can illustrate the quantity corresponding to $D_t(A|B)$ by bold straight line in Figure 3.

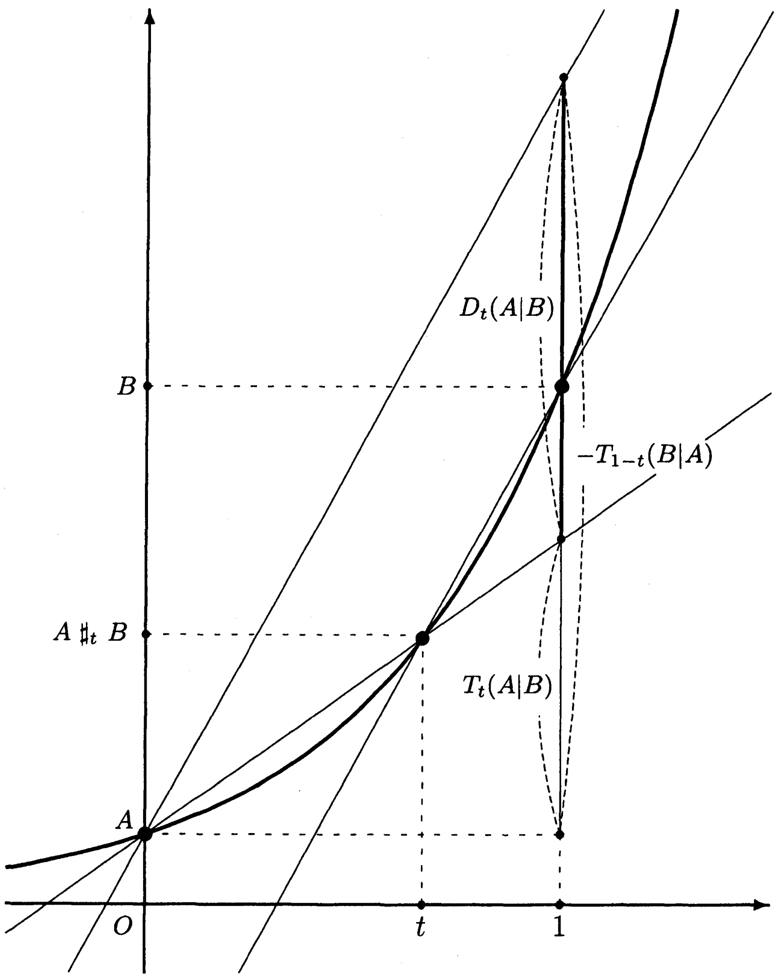

Figure 3: An interpretation for $D_t(A|B) = -T_{1-t}(B|A) - T_t(A|B)$.

Based on Theorem 4.1, we define expanded operator valued α-divergence.

Definition 4.2. *For strictly positive operators A and B, and for $t \in [0,1]$, $r \in [-1,1]$, expanded operator valued α-divergence is defined as follows:*

$$D_{t,r}(A|B) \equiv -T_{1-t,r}(B|A) - T_{t,r}(A|B).$$

Remark 3. *It is obvious that* $D_{1-t,r}(B|A) = D_{t,r}(A|B)$ *holds for* $t \in (0,1)$ *and* $r \in [-1,1]$.

We get the following relations for expanded operator valued α-divergence immediately.

Proposition 4.3. *Let* A *and* B *be strictly positive operators. Then,*

$$
\begin{array}{lrcl}
(1) & D_{t,0}(A|B) & = & D_t(A|B), \\
(2) & D_{t,1}(A|B) & = & 0, \\
(3) & D_{0,r}(A|B) & = & B - A - T_r(A|B), \\
(4) & D_{1,r}(A|B) & = & A - B - T_r(B|A)
\end{array}
$$

hold for $t \in [0,1]$ *and* $r \in [-1,1]$.

We can illustrate the relations in Proposition 4.3 as follows:

$$
\boxed{
\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow_{r=1} & & \\
B - A - T_r(A|B) & \underset{t=0}{\longleftarrow} & D_{t,r}(A|B) & \underset{t=1}{\longrightarrow} & A - B - T_r(B|A) \\
 & & \downarrow_{r\to 0} & & \\
 & & D_t(A|B) & &
\end{array}
}
$$

We can rewrite $D_{t,r}(A|B)$ as the difference between weighted arithmetic mean and operator power mean as follows.

Theorem 4.4. *Let* A *and* B *be strictly positive operators. Then,*

$$
D_{t,r}(A|B) = \frac{A \nabla_t B - A \sharp_{t,r} B}{t(1-t)}
$$

holds for $t \in (0,1)$ *and* $r \in [-1,1]$.

Proof. We can get this relation as follows:

$$
\begin{aligned}
D_{t,r}(A|B) &= -T_{1-t,r}(B|A) - T_{t,r}(A|B) = -\frac{B \sharp_{1-t,r} A - B}{1-t} - \frac{A \sharp_{t,r} B - A}{t} \\
&= \frac{-tA \sharp_{t,r} B + tB - (1-t)A \sharp_{t,r} B + (1-t)A}{t(1-t)} = \frac{A \nabla_t B - A \sharp_{t,r} B}{t(1-t)}.
\end{aligned}
$$

□

For expanded operator valued α-divergence, we are trying to obtain similar relations to Theorem 3.1 and Theorem 3.5. The followings are relations we have obtained until now.

Theorem 4.5. *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{llcl}
(1) & D_{t,r}(A|A \,\sharp_{s,r}\, B) & = & s\,\dfrac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t} \quad (t \in [0,1)), \\
(2) & D_{0,r}(A|A \,\sharp_{s,r}\, B) & = & s\left\{T_{s,r}(A|B) - T_r(A|B)\right\}, \\
(3) & D_{1,r}(A|A \,\sharp_{s,r}\, B) & = & s\left\{S_{s,r}(A|B) - T_{s,r}(A|B)\right\}.
\end{array}
$$

hold for $s \in (0,1)$ and $r \in [-1,1]$.

Proof. (1) By Theorem 3.5, we have

$$
\begin{aligned}
D_{t,r}(A|A \,\sharp_{s,r}\, B) &= -T_{1-t,r}(A \,\sharp_{s,r}\, B|A) - T_{t,r}(A|A \,\sharp_{s,r}\, B) \\
= s\left\{\frac{1}{1-t}T_{s,r}(A|B) - \frac{t}{1-t}T_{st,r}(A|B)\right\} - sT_{st,r}(A|B) &= s\,\frac{T_{s,r}(A|B) - T_{st,r}(A|B)}{1-t}.
\end{aligned}
$$

(2) We can get this result by putting $t = 0$ in (1).
(3) By Theorem 3.1 and Theorem 3.5, we have

$$
\begin{aligned}
& D_{1,r}(A|A \,\sharp_{s,r}\, B) = -T_r(A \,\sharp_{s,r}\, B|A) - T_{1,r}(A|A \,\sharp_{s,r}\, B) \\
= \; & S_{1,r}(A|A \,\sharp_{s,r}\, B) - T_{1,r}(A|A \,\sharp_{s,r}\, B) = sS_{s,r}(A|B) - sT_{s,r}(A|B) \\
= \; & s\left\{S_{st,r}(A|B) - T_{s,r}(A|B)\right\}.
\end{aligned}
$$

□

By the similar way to Theorem 4.5, we can obtain the results of operator valued α-divergence for fixed point A and any point on the path as follows:

Proposition 4.6. *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{llcl}
(1) & D_t(A|A \,\natural_s\, B) & = & s\,\dfrac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0,1)), \\
(2) & D_0(A|A \,\natural_s\, B) & = & s\left\{T_s(A|B) - S(A|B)\right\}, \\
(3) & D_1(A|A \,\natural_s\, B) & = & s\left\{S_s(A|B) - T_s(A|B)\right\}
\end{array}
$$

hold for $s \in \mathbf{R}$.

By applying noncommutative ratio translation to the relations in Proposition 4.6, we can get the results of operator valued α-divergence for any two points on the path as follows:

Theorem 4.7. *Let A and B be strictly positive operators. Then,*

$$
\begin{array}{llcl}
(1) & D_t(A \,\natural_v\, B|A \,\natural_{v+s}\, B) & = & s\mathcal{R}(v)\,\dfrac{T_s(A|B) - T_{st}(A|B)}{1-t} \quad (t \in [0,1)), \\
(2) & D_0(A \,\natural_v\, B|A \,\natural_{v+s}\, B) & = & s\mathcal{R}(v)\left\{T_s(A|B) - S(A|B)\right\}, \\
(3) & D_1(A \,\natural_v\, B|A \,\natural_{v+s}\, B) & = & s\mathcal{R}(v)\left\{S_s(A|B) - T_s(A|B)\right\}
\end{array}
$$

hold for $s,\, v \in \mathbf{R}$.

Proof. (1) Lemma 2.2, Proposition 4.6, and Theorem 2.9, we have

$$\begin{aligned} D_t(A \natural_v B | A \natural_{v+s} B) &= D_t(A \natural_v B | (A \natural_v B) \natural_s (A \natural_{v+1} B)) \\ = s\, &\frac{T_s(A \natural_v B | A \natural_{v+1} B) - T_{st}(A \natural_v B | A \natural_{v+1} B)}{1-t} \\ = s\mathcal{R}(v)\, &\frac{T_s(A|B) - T_{st}(A|B)}{1-t}. \end{aligned}$$

(2) We can get this result by putting $t = 0$ in (1).
(3) By (1) and (2) in Theorem 2.9,

$$\begin{aligned} D_1(A \natural_v B | A \natural_{v+s} B) &= A \natural_v B - A \natural_{v+s} B + S_1(A \natural_v B | A \natural_{v+s} B) \\ = S_1(A \natural_v B | A \natural_{v+s} B) - T_1(A \natural_v B | A \natural_{v+s} B) &= s\mathcal{R}(v) S_s(A|B) - s\mathcal{R}(v) T_s(A|B) \\ = s\mathcal{R}(v) \{S_s(A|B) - T_s(A|B)\}. & \end{aligned}$$

□

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